FUNCTIONS ARE AN IMPORTANT COMPONENT in the study of mathematics (NCTM 1989, 2000). Learning about the concept of functions can be a natural way for students to “mathematize” the real-world relationships that they observe. Everyday life abounds with opportunities for students to observe and describe dynamic relationships that can be classified as functions.

How can we describe understanding of functions in middle school students? At the middle school level, understanding functions involves understanding the dynamics of a relationship. A change in an input quantity is systematically related to a change in an output quantity. Recognizing the consistent relationship of input and output numbers suggests that a student “sees” beyond individual input-output values to focus on a systematic, generalized process (Breidenbach et al. 1992).

Students often first encounter functions in “input-output machines,” tables, and equations. They may report specific output values that are matched to specific input values or replace one variable with a numeric value and calculate the corresponding value for some other variable. At times, however, instruction is so focused on developing these procedural proficiencies that the consistency of the underlying relationship is overlooked.

How can students’ initial encounters with functions go beyond performance of isolated procedures? In a recent investigation, I explored the processes used by mixed-ability eighth-grade students when I introduced function concepts through dynamic physical models. The students had begun their study of algebra, but none had previously written equations to represent physical situations or used symbolic variables to refer to functions. Would ex-

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The Physical Models

In my models, students operate an input variable and observe an output variable, encountering functions in a nonsymbolic, real-world context. One of the dynamic physical models that my students used was a spool-elevating system (see fig. 1), similar to devices developed by Piaget and others (1977) to document the development of learners' knowledge of function. Greer (1994) also used a dynamic physical model to help students extend their understanding of multiplication and division of integers to operations with decimals.

The spool-elevating system consists of an arrangement of spools of varying circumferences, a small weighted object attached to one end of a cord that can be connected to any of the spools, and a ruler to measure the height of the object. I had the students work with the spool system because it concretely illustrates how two variables can change systematically. Students use the model to experience the variability of input and output values before using symbols to represent the variables or attempting to write equations.

Making a Spool-Elevating System

My SPool-Elevating SYSTEM WAS MANUFACTURED commercially, but I have also built a workable classroom model from household items. The directions for making the model were adapted from a pamphlet written by David Hendry (1998) on building physical models for classroom use. For each model you will need the following:

- A shoebox measuring about 6 inches wide, 3 to 4 inches deep, and 12 or more inches long
- A bendable drinking straw
- A large spool of sewing thread with enough thread removed to reduce the circumference to 4 inches (to keep the remaining thread from unwinding as the spool is operated, put a dot of glue on the end of the thread and secure it to the spool)
- A small spool of sewing thread with a circumference of 3 inches and all the thread removed
- An unsharpened pencil that will fit into the spools
- A paper clip or metal nut or washer for weight
- A ruler
- A hot-glue gun, masking tape, and transparent tape

To prepare the model, cut holes in the sides of the box near one end (see fig 1). Place the large spool onto the pencil, leaving 1 inch of the pencil visible on the eraser end. Glue the spool in place if needed. Slide the small spool onto the pencil next to the large spool, and glue it in place if necessary. Next to the small spool, wrap masking tape around the pencil until a third spool of 2-inch circumference has been built up. Snip the drinking straw so that a 1-inch length can be placed over the eraser.

Fig. 1 The spool-elevating system
end of the pencil to form a handle. Glue the straw handle onto the pencil. Attach a ruler to the inside wall of the box with transparent tape. Cut a 2-foot length of thread, and tie a paper clip at one end. Attach the other end with masking tape to any one of the spools, and begin the explorations.

**Explorations with the Spool-Elevating System**

STUDENTS CONTROL THE HEIGHT OF THE OBJECT on a particular spool by lengthening or shortening the cord through turns of the handle attached to the spool’s axle. For each spool, the number of handle turns and the height of the object are systematically related. By treating the number of handle turns as an input value to a function, the height of the object can be understood as an output value that varies according to, and is dependent on, the number of turns. For example, on the 3-spool, a spool with a circumference of 3 inches, a complete clockwise turn will raise the object 3 inches. Similarly, on the 4-spool, a clockwise turn will raise the object 4 inches. In an equation format, if \( n \) represents the number of handle turns and \( h \) represents the height of the object, then \( 3n = h \) for the 3-spool and \( 4n = h \) for the 4-spool.

I set up a learning station in a small conference room adjacent to the regular classroom. Working in pairs, the students freely explored the spool system without interruptions. I stayed with each pair of students and asked questions to prompt their thinking (see fig. 2). They answered most of the questions orally, but sometimes they wrote about, and drew pictures of, their thinking. I encouraged the students to first think individually about their answers, then to compare and discuss their ideas. I did not always ask the same questions of each pair of students. On the basis of their answers, I sometimes posed additional questions to learn more about the students’ thinking. The students noticed many variable and nonvariable features in the system. I asked them to think about the function relating the number of handle turns to the height of the object, in this case a metal mat, on a particular spool. They created symbols for these variables and made tables to explore the function.

The students generally observed that each complete clockwise handle turn consistently raised the object by the same amount. Some students described this relationship by repeatedly adding the same amount to the height of the object as the handle was turned. For example, as Jim explored the 2-spool, he reported, “If each crank goes up two inches and you want to go to nine, just count to nine. Two, four, six, eight . . . . It’d be four and a half [turns]. Just add two each time.” Other students indicated that the number of handle turns could be multiplied by a consistent amount to find the height. Dan, for example, filled in values in a table to ex-

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1. What happens when you place the object on a spool and operate the system?
2. How would you explain to someone how the spool system works?
3. What happens when the object is placed on different-sized spools?
4. Write a short paragraph explaining how the system works.
5. Draw a picture or diagram to show how the system works.
6. What aspects change as the system is operated?
7. What aspects stay the same as the system is operated?
8. Make up symbols to stand for two of the features that change (the number of turns of the handle or the height of the object) as the system is operated using the 3-spool.
9. How would someone who just walked into the room know what your symbols mean?
10. Even though your partner’s symbols are different, do they mean the same thing as yours? How do you know?
11. Make a table to show three possible values for each of the changeable characteristics in the system.
12. Have you listed all possible values for the changeable characteristics? Explain.
13. Even though your partner has different values in his or her table, do your tables mean the same thing?
14. Even though the individual values in your tables change, what feature always stays the same as you operate the system?
15. Use the symbols that you have created to write a mathematical sentence that tells how the changeable features work together.

**Fig. 2 Questions posed during explorations of the spool-elevating system**

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plore a similar function with the 3-spool. In the table, the input was given as whole numbers, but input values between 5 and 9 were omitted (see fig. 3). Dan multiplied 9 by 3 to get 27 and stated that he “jumped” from 5 to 9 without thinking about the values between 15 and 27. Using the same table presented to Dan, Hilaria lightly dashed in the input values 6, 7, and 8 (see fig. 4). She then computed values for the function by counting groups of three on the fingers of one hand as she kept track of the number of groups on the other hand.

Jim and Hilaria viewed the functions in terms of repeated addition. They needed to compute each value for the function in sequence without skipping any values. Their thoughts were focused on the changes in values in the output column of the table. Dan understood the function as multiplication, which he used to find values in the function without working in sequence. The focus of his thinking was on how the input was related systematically to the output.

Students sometimes became confused as they thought about the functions, such as when Claudia created a table to display the data for different-sized spools (see fig. 5). She noticed that for every spool, the heights of the object after two cranks of the handle were double the heights after one crank; thus, she decided that the functions were multiplying by two. When she concentrated on filling in values for the 3-spool, she expected each output value to be double the previous value and could not explain the discrepancies that she observed from operating the spool system. As Jim and Hilaria had done, Claudia focused on the changes in output values and did not notice the relationship between input and output. She confused the varying number of handle cranks, which was two at the time she decided the function was doubling, with the constant amount of elevation for each crank, which was three when she considered the 3-spool function.

Writing Mathematical Sentences

WHEN STUDENTS BEGAN TO USE SYMBOLS TO write mathematical sentences representing the functions in the spool system, many of the same interpretations occurred. The students created mathematical sentences to preserve the systematic operation of the system. Some of the sentences, such as Robert’s, suggested a repeated-addition interpretation (see fig. 6). Other mathematical sentences, such as one written by Dan to represent the function given in a table, indicated a multiplication view (see fig. 7).

As Dan did in his equation, several students wrote equations in which the input variable for the number of turns of the spool was given as the leftmost symbol. That symbol was followed by multiplication by a number corresponding to the height of
the object for each turn of the spool handle. The equations were completed with an equals sign followed by a symbol to represent the height of the object, for example, $x \times 2 = h$. When I rearranged the equations to the format commonly used in textbooks, $2 \times x = h$ or $2x = h$, the meanings of the mathematical sentences changed for these students. Even though they had studied the commutative property, the students interpreted the symbols in the rearranged mathematical sentences differently when the sentence was intended to represent a function in the spool system. For these students, the leftmost symbol always represented the variable number of handle turns, regardless of whether the symbol was a symbolic variable or a numeral.

These students also treated the symbol following the multiplication sign, whether it was a symbolic variable or a numeral, as a constant representing the increase in elevation with each turn of the handle.

To these students, a mathematical sentence such as $3 \times x = h$ indicated that three handle turns are taken and that $x$ is the increase in elevation for each handle turn. The left-to-right sequence of symbols in a mathematical sentence that represented a function in the spool system needed to correspond to the action of the model as the students perceived it. Not only were the meanings of the symbols interpreted more broadly than their originally assigned individual meanings, but the location of the symbols in the sentence also contributed to their meaning as a representation of a function in the spool system.

**Conclusion**

**STUDENTS NEED PLENTY OF TIME TO DISCUSS**

and distinguish among variables and nonvariables and to experience the systematic relationship between the variables in the functions of the spool system. Once students are familiar with the spool system, this activity can be extended to include comparing functions generated on different-sized spools. In this way, students can develop a deeper appreciation of the individual functions as unified, systematic processes. They can also observe the effect of different rates of change for different-sized spools and can consider the way that different rates appear in the tables and mathematical sentences representing the functions. Students can further explore rates of change by preparing and exploring graphs that compare spool sizes with the different heights attained as the object is elevated by turns of the handle.

**References**


